SOLUTION OF NONLINEAR HEAT-CONDUCTION PROBLEMS FOR A SEMITRANSPARENT BODY

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We solve the nonlinear heat-transfer problem in a semitransparent optically thick body by the finite-differences method in combination with the small-parameter method.

The heat-transfer processes in nonlinear media have been given considerable attention in recent years. Investigators are generally familiar with the heat-conduction equations related to high-temperature processes when the thermophysical properties of the medium vary with the temperature. However, the nonlinearity may also be due to internal causes and attributable to the properties of the material.

In the present study we consider the thermal field of a semitransparent, hard, optically thick body with a direct current fed to its surface. Part of the current is reflected from the surface of the body, while the remaining part, passing through the body, is continuously attenuated because of internal absorption by the material. Radiant transfer of energy takes place within the body. This effect leads to the appearance of a nonlinear term in the heat-balance equation [1, 2].

The temperature inside the body will satisfy the equation

$$c\rho\,\frac{\partial T}{\partial t}=\operatorname{div}(\lambda\operatorname{grad}T+F).$$

Here c, ρ, λ are the heat capacity, density, and thermal conductivity of the medium, respectively; T, temperature; t, time; F, radiant-energy flux.

$$F=\frac{4\sigma}{3\alpha_\rho}\operatorname{grad}T^4,$$

where σ is the Stefan–Boltzmann constant; α_ρ is the coefficient of absorption of the radiation.

I. Finite-Difference Method. Suppose that the body is a plate of finite thickness and unbounded length:

$$\frac{\partial T}{\partial t}=a^2\,\frac{\partial^2 T}{\partial x^2}+\alpha^2\,\frac{\partial^2 T^4}{\partial x^2},\tag{1}$$

$$T(x,\ 0)=0,\ T_x(0,\ t)=0,\ -T_x(l,\ t)+\frac{q}{l}=0,\tag{2}$$

q is the constant flux on the external surface; $a^2 = \lambda/c\rho$; $\alpha^2 = 4\sigma/3c\rho\alpha_\rho \ll 1$.

We construct the difference system for problem (1)-(2):

$$\tau^{-1}[U_{j,k+1}-U_{jk}]=a^2h^{-2}[U_{j+1,k+1}-2U_{j,k+1}+U_{j-1,k+1}]+$$
$$+h^{-2}[4\alpha^2U_{jk}(U_{j+1,k+1}-U_{j,k+1})-4\alpha^2U_{j-1,k}(U_{j,k+1}-U_{j-1,k+1})],\tag{3}$$

$$U_{j0}=0;\ U_{0k}=U_{1k};\ U_{Nk}-U_{N-1,k}=hql^{-1};\ j=0,\ 1,\ \ldots,\ N;\quad k=0,\ 1,\ \ldots\tag{4}$$

This scheme approximates problem (1), (2) to the order O(τ + h), where h is the step along the x axis and τ is the step along the t axis. Let U^h be the grid function. We write

$$R_hU^h\equiv(a^2+4\alpha^2U_{jk}^3)\,\tau h^{-2}U_{j+1,k+1}+(a^2+4\alpha^2U_{j-1,k}^3)\,\tau h^{-2}U_{j-1,k+1}-$$
$$-[1+2a^2\tau h^{-2}-4\alpha^2\tau h^{-2}(U_{jk}^3+U_{j-1,k}^3)]\,U_{j,k+1}+U_{jk}=0,\tag{5}$$

$$l_1U^h\equiv U_{0k}-U_{1k}=0,\ l_2U^h\equiv U_{Nk}-U_{N-1,k}=hql^{-1}.\tag{6}$$

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We make use of the Babenko–Gel'fand criterion [3]: problem (5)-(6) is stable if the set of eigenvalues of each of the following three problems:

$$R_h U^h = 0,\ U_{j0} = 0,\ j = 0,\ \pm 1,\ \pm 2,\ \ldots;\ U_{jk} \xrightarrow[j \to \pm\infty]{} 0 \tag{7}$$

(the coefficients for $U_{j+1,k+1}$, $U_{j,k+1}$, $U_{j-1,k+1}$ are frozen at an arbitrary interior point (x, t), $0 < x < l$, $0 < l < T$),

$$R_h U^h = 0,\ l_1 U^h = 0,\ j = 0,\ 1,\ \ldots;\ U_{jk} \xrightarrow[j \to +\infty]{} 0 \tag{8}$$

(the coefficients for $U_{j+1,k+1}$, $U_{j,k+1}$, $U_{j-1,k+1}$ are frozen at the point (0, t)),

$$R_h U^h = 0,\ l_2 U^h = 0,\ j = 0,\ -1,\ -2,\ \ldots;\ U_{jk} \xrightarrow[j \to -\infty]{} 0 \tag{9}$$

(the coefficients for $U_{j+1,k+1}$, $U_{j,k+1}$, $U_{j-1,k+1}$ are frozen at the point (l, t)) lies in the unit disk.

Investigating problem (7), we find the spectrum of the difference operator, passing from layer t_k to layer t_{k+1} at the interior points of the region, disregarding the boundary conditions.

In investigating problem (8), we take account of the effect of the left-hand boundary condition but disregard the right-hand condition, and vice versa in the case of (9).

1. We consider the problem (7). We shall try to find a solution of this equation in the form

$$U_{jk} = \lambda^k Z^j. \tag{10}$$

Substituting (10) into (7), we obtain

$$A\lambda^{k+1} Z^{j+1} - B\lambda^{k+1} Z^j + C\lambda^{k+1} Z^{j-1} + \lambda^k Z^j = 0.$$

Here

$$A = \tau h^{-2}(a^2 + 4\alpha^2 U_{jk}^3);\ C = \tau h^{-2}(a^2 + 4\alpha^2 U_{j-1,k}^3);\ B = 1 + A + C. \tag{11}$$

Suppose that the coefficients A, B, C are frozen at an arbitrary interior point. We find Z:

$$\lambda[AZ^2 - BZ + C] = -Z,\ AZ^2 - (B - \lambda^{-1})Z + C = 0,$$

$$Z_{1,2} = \frac{1}{2A}[B - \lambda^{-1} \pm \sqrt{(B - \lambda^{-1})^2 - 4AC}]. \tag{12}$$

If $Z = Z_1$, at $Z = Z_2$, then (10) is the solution of problem (7).

Function $\lambda^h Z^j$ can remain bounded only when $|Z| = 1$, i.e., $Z = \exp(i\varphi)$. Therefore, the solution of problem (7) must be sought in form $\lambda^k \exp(i\varphi)$, where it follows from (11) that

$$\frac{1}{\lambda} = 1 + 4a^2 \tau h \sin\frac{\varphi}{2} - 4\alpha^2 \tau h^{-2}[U_{jk}^3(e^{i\varphi} - 1) + U_{j-1,k}^3(e^{-i\varphi} - 1)].$$

Since α is small by hypothesis, it follows that $|\lambda| < 1$ for any τ, h.

2. We consider problem (8). Its solution must satisfy the condition $U_{jk} \xrightarrow[j \to +\infty]{} 0$ and consequently has the form gZ_2^k, $|Z_2| < 1$:

$$U_{jk} = g(2A)^{-1}[B - \lambda^{-1} - \sqrt{(B - \lambda^{-1})^2 - 4AC}].$$

The value of λ must be found in such a way that U_{jk} satisfies the boundary condition

$$l_1 U^h \equiv U_{1k} - U_{0k} = 0,\ g(Z - 1) = 0.$$

But $|Z| \neq 1$; consequently, $g = 0$ and problem (8) has no eigenvalues.

3. We consider problem (9). We shall try to find its solution in the form $U_{jk} = gZ_1^k$. By a reasoning analogous to the previous case, we see that problem (9) has no eigenvalues.

Thus, the set of eigenvalues of problems (7)-(9) lies in the unit disk. By the Babenko–Gel'fand criterion, problem (3)-(4) is stable.

II. Small-Parameter Method. In investigating a process it is often necessary to know how the nonlinearity on the right side affects the solution and how the solution of the linear equation and the solution of the equation with the added nonlinear term are related. In our case we are interested in establishing the effect of the radiant-energy source $(4\sigma/3\alpha_\rho)\mathrm{grad}\,T^4$ on the thermal state of the body.

To do this, we used the small-parameter method. It is convenient to use a dimensionless quantity as the parameter. Therefore, we reduce (1)-(2) to the dimensionless form

$$\frac{\partial V}{\partial \mathrm{Fo}} = \frac{\partial^2 V}{\partial y^2} + \mu \frac{\partial^2 (1+V)^4}{\partial y^2}, \qquad (13)$$

$$V(y,\ 0)=0,\ V_y|_{y=0}=0,\ V_y|_{y=1}=\mathrm{Ki}$$

by means of the change of variables

$$\mathrm{Fo}=\frac{a^2 t}{l},\ y=\frac{x}{l},\ T_0=293^\circ\mathrm{K},\ V=\frac{T-T_0}{T_0},$$

$$\mu=\frac{4\sigma T_0^3}{3a^2 c\rho\alpha_\rho},\ \mathrm{Ki}=\frac{ql}{a^2 c\rho T_0}.$$

The solution of problem (13) will be sought in the form of a series in powers of the parameter μ:

$$V=V_1+\mu V_2+\mu^2 V_3+\ \ldots \qquad (14)$$

Substituting (14) into (13), we compare the coefficients for equal powers of μ and obtain the sequence of solutions

$$\frac{\partial V_1}{\partial \mathrm{Fo}}=\frac{\partial^2 V_1}{\partial y^2},$$

$$\frac{\partial V_2}{\partial \mathrm{Fo}}=\frac{\partial^2 V_2}{\partial y^2}+12(1+V_1)^2 (V'_{1y})^2+4(1+V_1)^3\frac{\partial^2 V_1}{\partial y^2}, \qquad (15)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial V_n}{\partial \mathrm{Fo}}=\frac{\partial^2 V_n}{\partial y^2}+f(V_1,\ \ldots,\ V_{n-1},\ V'_{1y},\ \ldots,\ V'_{(n-1)y}).$$

We solve the first equation for the original initial and boundary conditions, and all the other equations for zero conditions. The solution of the problem

$$\frac{\partial V_1}{\partial \mathrm{Fo}}=\frac{\partial^2 V_1}{\partial y^2}, \qquad (16)$$

$$V_1(y,\ 0)=0,\ V'_{1y}|_{y=0}=0,\ V'_{1y}|_{y=1}=\mathrm{Ki}$$

by the finite-difference method can be carried out without difficulty, and we shall not discuss it further.

The function V_2 is a solution of the problem

$$\frac{\partial V_2}{\partial \mathrm{Fo}}=\frac{\partial^2 V_2}{\partial y^2}+12(1+V_1)^2 (V'_{1y})^2+4(1+V_1)^3\frac{\partial^2 V_1}{\partial y^2}, \qquad (17)$$

$$V_2(y,\ 0)=V'_{2y}|_{y=0}=V'_{2y}|_{y=1}=0.$$

Using the grid $y_j = jh$, $(\mathrm{Fo})_k = k\tau$, we construct a difference scheme approximating problem (17):

$$(U_{j,k+1}-U_{jk})\,h^2\tau^{-1}=U_{j+1,k+1}-2U_{j,k+1}+U_{j-1,k+1}+h^2 f^{(2)}_{j,k+1}, \qquad (18)$$

$$U_{j0}=0,\ U_{0k}=U_{1k},\ U_{Nk}=U_{N-1,k}.$$

Here

$$h^2 f^{(2)}_{j,k+1}=12[1+V_{1;j,k+1}]^2[V_{1;j+1,k+1}-V_{1;j,k+1}]+4[1+V_{1;j,k+1}]^3[V_{1;j+1,k+1}-2V_{1;j,k+1}+V_{1;j-1,k+1}],$$

$f^{(2)}_{j,k+1}$ is a known function, since the values of V_1 at the points of the grid have already been found from (16). It is not difficult to see that all the eigenvalues λ of the difference operator of the linear problem (18) lie in the unit disk:

$$\lambda=\left(1+4a^2\tau h^{-2}\sin\frac{\varphi}{2}\right)^{-1}.$$

Consequently, scheme (18) is stable.

After obtaining V_2, we proceed to solve the third equation, etc. The difference scheme for the n-th approximation has the form

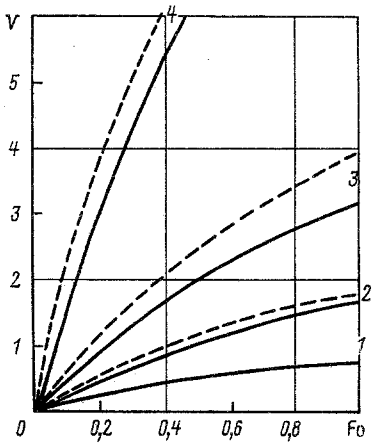

Fig. 1. Dimensionless temperature V as a function of dimensionless time Fo for values of the criterion Ki = 0.5 (1); 1.0 (2); 2.0 (3); 3.0 (4) for problem (19)-(20). The solid curves indicate the solution of the problem by the small-parameter method with two terms of the expansion (15), and the dashed curves represent the solution of the problem obtained by method of simulation on the R grid analog.

$$(U_{j,k+1} - U_{jk})\,\tau^{-1}h^2 = U_{j+1,k+1} - 2U_{j,k+1} + U_{j-1,k+1} + h^2 f^{(n)}_{j,k+1},$$
$$U_{j0} = 0,\ U_{0k} = U_{1k},\ U_{Nk} = U_{N-1,k}.$$

This scheme differs from (18) only in the form of the free term, which is a function of the previously found quantities $V_1, \ldots, V_{n-1}, (V_1)_y, \ldots, (V_{n-1})_y$. If in the expansion in terms of the parameter we take, e.g., three terms, then the calculation can be conveniently carried out as follows:

1) find the values of V_1 on the first time layer $Fo = \tau$;

2) calculate the values of $f^{(2)}_{j1}$ on the layer $Fo = \tau$;

3) find the solution of problem (18), i.e., the values of V_2, on the layer $Fo = \tau$;

4) calculate the values of the right-hand side $f^{(3)}_{j1}$ of the following difference problem on the layer $Fo = \tau$;

5) find the approximate values of V_3 on the layer $Fo = \tau$;

6) set up $V_1 + \mu V_2 + \mu^2 V_3$. We obtain an approximate solution of problem (1)-(2) at the points of the grid for $Fo = \tau$;

7) repeat operations 1)-6) for $Fo = 2\tau, 3\tau, \ldots$

We shall show how the above methods are used in solving the following heat-transfer problem in an optically thick plate:

$$\frac{\partial V}{\partial \mathrm{Fo}} = \frac{\partial}{\partial x}\left[(1 + 4\mu(1 + V)^3 \frac{\partial V}{\partial x}\right],\quad 0 < x < 1; \tag{19}$$

$$V(x,\ 0) = 0,\ \left.\frac{\partial V}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial V}{\partial x}\right|_{x=1} = \mathrm{Ki}; \tag{20}$$

$$\mu = 4\sigma T_0^3/3\lambda\alpha_\rho,\ \sigma = 5.69\cdot 10^{-8}\,\mathrm{W/m^2\cdot(deg\ K)^4},\ T_0 = 293\,{}^\circ\mathrm{K},$$
$$\alpha_\rho = 100\,\mathrm{m^{-1}},\ \lambda = 1.185\,\mathrm{W/m\cdot deg\ K},\ \mathrm{Ki} = 0.5;\ 1.0;\ 2.0;\ 3.0,$$

V is the dimensionless temperature.

The solution of this problem by the small-parameter method with two terms in the expansion (15) is shown in Fig. 1 (solid curves). The graphs are taken from [1]. As the small parameter we took $\mu = 0.0161$. Figure 1 also shows the graphs of the solution of problem (19)-(20) (dashed curves), which were obtained by the method of simulation on the R grid analog. We used the implicit difference scheme (5) written in the form

$$\frac{h}{\tau}[V_{i,k+1}-V_{ik}]=[1+4\mu(1+V_{i+1,k})^3][V_{i+1,k+1}-V_{i,k+1}]-$$
$$-[1+4\mu(1+V_{i,k})^3][V_{i,k+1}-V_{i-1,k+1}],\ i=1,\ \ldots,\ N-1;$$
$$V_{i0}=0,\ V_{0k}=V_{1k},\ V_{Nk}-V_{N-1,k}=h\,\mathrm{Ki}. \tag{21}$$

We selected $h = 0.1$, $\tau = 0.05$.

It can be seen from the graphs that for $0 \le \mathrm{Ki} \le 1$ the temperature fields calculated by the small-parameter method with two terms of the expansion (15) agree fairly well with the solution of the problem (21).

It was established experimentally that for $2 \le \mathrm{Ki} \le 3$ taking account of the third term of the expansion in powers of the small parameter in (15) yields a temperature field which practically coincides with the fields calculated from (21).

The third term is determined from the conditions:

$$\frac{\partial V_3}{\partial \mathrm{Fo}}=\frac{\partial^2 V_3}{\partial x^2}+f(V_1,\ V_2),$$
$$V_3(x,\ 0)=0,\ V'_{3x}|_{x=0}=V'_{3x}|_{x=1}=0, \tag{22}$$
$$f=12(1+V_1)^2[2V'_{1x}V'_{2x}+V''_{1xx}]+4(1+V_1)^3V''_{2xx}+24V_2(1+V_1)[V'_{1x}]^2.$$

The values of V_1 and V_2 are taken from [1].

Equation (22) was also simulated on the R grid analog. We used the difference scheme of (18), in the form

$$\frac{V_{i,k+1}-V_{ik}}{\tau}=\frac{V_{i+1,k+1}-2V_{i,k+1}+V_{i-1,k+1}}{h^2}+f_{i,k+1},$$
$$V_{i0}=0,\ V_{0k}=V_{1k},\ V_{Nk}=V_{N-1,k}.$$

LITERATURE CITED

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